Double points of paths of Brownian motion in n-space.

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§ 1. Introduction.

Let $(\Omega, \mathcal{E}, \Pr)$ be a probability space, i. e. $\Omega=\{\omega\}$ is a set of elements ω, $\mathcal{E}=\{E\}$ is a Borel field of subsets E of Ω called "events", and $\Pr(E)$ is a countably additive measure defined on $\mathcal{E}$ with the normalization $\Pr(\Omega)=1$. $\Pr(E)$ is called the "probability" of the event E.

A *one-dimensional Brownian motion* [cf. **3**, **5**, **6**, **7**] is a real-valued function $x(t, \omega)$ of the two variables t and ω, defined for all non-negative real numbers t, $0 \leqq t < \infty$, and for all $\omega \in \Omega$, with the following properties:

(B_1) $x(0, \omega) \equiv 0$,

(B_2) for any real numbers s, t with $0 \leqq s < t < \infty$, $x(t, \omega)-x(s, \omega)$ is $\mathcal{E}$-measurable in ω and has a Gaussian distribution with mean value 0 and variance $t-s$, i. e. [1])

$$E_{x, s, t, \alpha, \beta} \equiv \{\omega \mid \alpha < x(t, \omega)-x(s, \omega) < \beta\} \in \mathcal{E}, \tag{1}$$

and

$$\Pr(E_{x, s, t, \alpha, \beta}) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\alpha}^{\beta} e^{-\frac{u^2}{2(t-s)}}\, du \tag{2}$$

for any real numbers α, β with $-\infty < \alpha < \beta < \infty$,

(B_3) for any real numbers s_k, t_k $(k=1, \dots, p)$ with $0 \leqq s_1 < t_1 \leqq s_2 < t_2 \leqq \leqq \dots \leqq s_p < t_p < \infty$, the functions $x(t_k, \omega)-x(s_k, \omega)$, $k=1, \dots, p$, are independent in the sense of probability theory, i. e.

$$\Pr\left(\bigcap_{k=1}^{p} E_{x, s_k, t_k, \alpha_k, \beta_k}\right) = \Pi_{k=1}^{p} \Pr(E_{x, s_k, t_k, \alpha_k, \beta_k}) \tag{3}$$

for any real numbers α_k, β_k with $-\infty < \alpha_k < \beta_k < \infty$, $k=1, \dots, p$.

An *n-dimensional Brownian motion* is an n-system of mutually independent one-dimensional Brownian motions, i. e. an n-system $\{x^i(t, \omega) \mid i=1, \dots, n\}$ of one-dimensional Brownian motions $x^i(t, \omega)$, $i=1, \dots, n$, with the property that

$$\Pr\left(\bigcap_{i=1}^{n} E^i\right) = \Pi_{i=1}^{n} \Pr(E^i), \tag{4}$$

where E^i is any subset of Ω determined by $x^i(t, \omega)$, i. e. a subset of Ω which belongs to the Borel subfield $\mathcal{E}^i$ of $\mathcal{E}$ which is generated by $\{E_{x, s, t, \alpha, \beta} \mid 0 \leqq s < t < \infty, \quad -\infty < \alpha < \beta < \infty\}$, $i=1, \dots, n$.

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[1]) $\{\omega \mid \dots\}$ denotes the set of all ω having the properties ..., and similarly in other cases.

If we consider $\mathbf{x}(t, \omega) = \{x^i(t, \omega) \mid i = 1, \ldots, n\}$ as a point in an n-dimensional Euclidean space R^n, then, for each fixed ω, $\mathbf{x}(t, \omega)$ can be considered as an R^n-valued function of t defined for $0 \leqq t < \infty$.

It is easy to see that this definition of an n-dimensional Brownian motion is independent of the choice of the rectangular coordinate system, i. e. it is invariant vis-à-vis rotations of the coordinate system.

It is assumed (cf. DOOB [1]) that the Borel field $\mathcal{E}$ is already extended by adding null sets in such a way that the subset C of Ω consisting of all ω for which $\mathbf{x}(t, \omega)$ is a continuous function of t for $0 \leqq t < \infty$ is $\mathcal{E}$-measurable and satisfies $\Pr(C) = 1$.

For any $\mathbf{y} = \{y^1, \ldots, y^n\} \in R^n$ and for any $\omega \in \Omega$, let us put

(5) $L^{(n)}_{a,b}(\mathbf{y}; \omega) = \{\mathbf{y} + \mathbf{x}(t, \omega) \mid a \leqq t \leqq b\}, \quad 0 \leqq a < b < \infty,$

(6) $L^{(n)}_{a,\infty}(\mathbf{y}; \omega) = \{\mathbf{y} + \mathbf{x}(t, \omega) \mid a \leqq t < \infty\}, \quad 0 \leqq a < \infty,$

(7) $L^{(n)}(\mathbf{y}; \omega) = L^{(n)}_{0,\infty}(\mathbf{y}; \omega),$

(8) $L^{(n)}_{a,b}(\omega) = L^{(n)}_{a,b}(\mathbf{0}; \omega), \quad L^{(n)}_{a,\infty}(\omega) = L^{(n)}_{a,\infty}(\mathbf{0}; \omega), \quad L^{(n)}(\omega) = L^{(n)}(\mathbf{0}; \omega),$

where $\mathbf{y} + \mathbf{x}(t, \omega) = \{y^i + x^i(t, \omega) \mid i = 1, \ldots, n\}$. $L^{(n)}_{a,b}(\mathbf{y}; \omega)$ is called the *(a, b)-path* of the n-dimensional Brownian motion starting from $\mathbf{y}$ and $L^{(n)}(\mathbf{y}; \omega)$ is called the *path* of the n-dimensional Brownian motion starting from $\mathbf{y}$.

For almost all ω (i. e. for all $\omega \in C$), $L^{(n)}_{a,b}(\mathbf{y}; \omega)$ is a continuous image of a closed interval $[a, b] = \{t \mid a \leqq t \leqq b\}$, and is hence a compact subset of R^n.

$\mathbf{x}_0 = \{x^1_0, \ldots, x^n_0\} \in R^n$ is called a double point of $L^{(n)}_{a,b}(\mathbf{y}; \omega)$ [resp. of $L^{(n)}_{a,\infty}(\mathbf{y}; \omega)$], if there exists a pair of real numbers s, t with $a \leqq s < t \leqq b$ [resp. $a \leqq s < t < \infty$] such that $\mathbf{x}_0 = \mathbf{y} + \mathbf{x}(s, \omega) = \mathbf{y} + \mathbf{x}(t, \omega)$ (i. e. $x^i_0 = y^i + x^i(s, \omega) = y^i + x^i(t, \omega)$, $i = 1, \ldots, n$). It is clear that $\mathbf{x}_0$ is a double point of $L^{(n)}_{a,b}(\mathbf{y}; \omega)$ [resp. $L^{(n)}_{a,\infty}(\mathbf{y}; \omega)$] if and only if $\mathbf{x}_0 - \mathbf{y}$ is a double point of $L^{(n)}_{a,b}(\mathbf{0}; \omega) = L^{(n)}_{a,b}(\omega)$ [resp. $L^{(n)}_{a,\infty}(\mathbf{0}; \omega) = L^{(n)}_{a,\infty}(\omega)$].

It is known that (i) [LÉVY 6] in R^2, almost all paths $L^{(2)}(\omega)$ of a 2-dimensional Brownian motion have double points and (ii) [3] in R^5, almost all paths $L^{(5)}(\omega)$ of a 5-dimensional Brownian motion have no double points. (ii) evidently implies that almost all paths in R^n with $n \geqq 5$ have no double points. Thus the problem of double points of paths of an n-dimensional Brownian motion is unsettled only for the cases $n = 3, 4$. These cases do not yield to the methods used in proving (i) and (ii); it is the purpose of this paper to dispose of these undecided cases by showing that (iii) in R^3, almost all paths $L^{(3)}(\omega)$ have double points, while (iv) in R^4, almost all paths $L^{(4)}(\omega)$ have no double points.

The proof of these results will be given in § 3 and § 4 respectively.

Our proof is based on the notion of capacity which plays an important role in the theory of harmonic functions in R^n. The definition of capacity and the statement of those of its fundamental properties which we need in the proofs of § 3 and § 4 will be found in § 2.

§ 2. Capacity.

Let F be a compact subset of $R^n (n \geqq 3)$. Let $\mathfrak{M}(F)$ be the family of all countably additive measures $m(B)$ defined for all Borel subsets B of F with $m(F)=1$. Let us put

$$\lambda^{(n)}(F) = \inf \iint \frac{m(d\mathbf{x})\, m(d\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}}, \tag{9}$$

where $|\mathbf{x}|$ denotes the distance of $\mathbf{x}$ from the origin $\mathbf{0}$ of R^n, so that $|\mathbf{x}-\mathbf{y}|$ is the distance of $\mathbf{x}$ and $\mathbf{y}$ in R^n; the double integral is extended over $F \times F$, and inf denotes the infimum for all measures $m \in \mathfrak{M}(F)$. $\lambda^{(n)}(F)=\infty$ if and only if the double integral is ∞ for all $m \in \mathfrak{M}(F)$. The n-dimensional capacity $C^{(n)}(F)$ of F is defined by

$$C^{(n)}(F) = \begin{cases} [\lambda^{(n)}(F)]^{-\frac{1}{n-2}} & \text{if } \lambda^{(n)}(F) < \infty, \\ 0 & \text{if } \lambda^{(n)}(F) = \infty. \end{cases} \tag{10}$$

The notion of capacity is important in the theory of harmonic functions in R^n, where under a harmonic function $f(\mathbf{x})$ defined in a domain D of R^n we understand a real-valued function $f(\mathbf{x})$ with continuous second partial derivatives which satisfies

$$\Delta f(\mathbf{x}) \equiv \sum_{i=1}^{n} \left(\frac{\partial}{\partial x^i}\right)^2 f(\mathbf{x}) \equiv 0 \tag{11}$$

in D.

In this paper we need the following properties of the capacity:

(C_1) [FROSTMAN 2] Let $F = \{\mathbf{x}(t) | a \leqq t \leqq b\} \subset R^n$ be the continuous image of a closed interval $[a, b] = \{t \mid a \leqq t \leqq b\}$ of real numbers through the mapping $t \to \mathbf{x}(t)$. (This mapping need not be one-to-one.) Then the n-dimensional capacity of F is positive if

$$\int_a^b \int_a^b \frac{ds\,dt}{|\mathbf{x}(t)-\mathbf{x}(s)|^{n-2}} < \infty. \tag{12}$$

(C_2) [PÓLYA—SZEGŐ 9] For any compact subset F of R^n, let us put

$$\lambda_p^{(n)}(F) = \inf \frac{2}{p(p-1)} \sum_{1 \leqq i < j \leqq p} \frac{1}{|\mathbf{x}_j - \mathbf{x}_i|^{n-2}}, \tag{13}$$

where inf denotes the infimum for all p-systems $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} \subset F$. Then

$$\lim_{p \to \infty} \lambda_p^{(n)}(F) = \lambda^{(n)}(F). \tag{14}$$

(C_3) [9] The union of a finite number of compact subsets of R^n each of which has zero n-dimensional capacity has again zero n-dimensional capacity.

(C_4) [2] In order that a compact subset F of R^n have positive n-dimensional capacity, it is necessary and sufficient that there exist a function $g(\mathbf{y})$ harmonic, positive and smaller than 1 in R^n-F, and satisfying $g(\mathbf{y}) \to 0$ as $|\mathbf{y}| \to \infty$.

We need also the following result:

L e m m a 1. *Let F be a compact subset of R^n $(n \geqq 3)$. For any $\mathbf{y} \in R^n - F$ let* *us put* $\Omega(\mathbf{y}; F)=\{\omega \mid L^{(n)}(\mathbf{y};\omega)\cap F\neq\theta\}$.[2]) *Then* $\Omega(\mathbf{y}; F)\in\mathcal{E}$ *and* $\Pr[\Omega(\mathbf{y};F)]=f(\mathbf{y};F)$ *is a harmonic function of* $\mathbf{y}$ *defined in* R^n-F. *Furthermore,* (i) $f(\mathbf{y};F)\equiv 0$ *in* R^n-F *if* $C^{(n)}(F)=0$; (ii) $0<f(\mathbf{y};F)<1$ *in* R^n-F; *and* $f(\mathbf{y};F)\to 0$ *as* $|\mathbf{y}|\to\infty$ *if* $C^{(n)}(F)>0$.

In the two-dimensional case the situation is rather different: (i) is still valid, but if the two-dimensional (logarithmic) capacity[3]) of F is positive then $f(\mathbf{y};F)\equiv 1$. This result can be found in [4] and the method of proof used there yields also our Lemma 1 for $n\geq 3$. This is due to the property (C_4) of the capacity which holds only for $n\geq 3$.

§ 3. The 3-dimensional case.

Lemma 2. *Let* $0\leq a<b<\infty$. *Then, for almost all* ω, *the* (a,b)*-path* $L^{(3)}_{a,b}(\omega)$ *of a 3-dimensional Brownian motion has positive 3-dimensional capacity.*

Proof. Due to property (C_1) of the capacity, it suffices to show that

$$\int_a^b\int_a^b \frac{ds\,dt}{|\mathbf{x}(t,\omega)-\mathbf{x}(s,\omega)|}<\infty \tag{15}$$

for almost all ω, and hence it suffices to show that

$$I=\int_\Omega d\omega\int_a^b\int_a^b \frac{ds\,dt}{|\mathbf{x}(t,\omega)-\mathbf{x}(s,\omega)|}<\infty. \tag{16}$$

It is easy to see [by (B_2) and (B_3) of § 1] that

$$\begin{aligned}\int_\Omega \frac{d\omega}{|\mathbf{x}(t,\omega)-\mathbf{x}(s,\omega)|}&=\left(\frac{1}{\sqrt{2\pi|t-s|}}\right)^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{u^2+v^2+w^2}{2|t-s|}\right)}{\sqrt{u^2+v^2+w^2}}\,du\,dv\,dw=\\ &=\left(\frac{1}{\sqrt{2\pi|t-s|}}\right)^3\int_0^\infty\frac{\exp\left(-\frac{r^2}{2|t-s|}\right)}{r}\cdot 4\pi r^2\,dr=\left(\frac{1}{\sqrt{2\pi|t-s|}}\right)^3\cdot 4\pi|t-s|=\sqrt{\frac{2}{\pi|t-s|}}\end{aligned} \tag{17}$$

and consequently, by the Fubini theorem,

$$I=\int_a^b\int_a^b ds\,dt\int_\Omega\frac{d\omega}{|\mathbf{x}(t,\omega)-\mathbf{x}(s,\omega)|}=\sqrt{\frac{2}{\pi}}\int_a^b\int_a^b\frac{ds\,dt}{\sqrt{|t-s|}}<\infty. \tag{18}$$

We can now prove our first main result:

Theorem 1. *In a 3-dimensional Brownian motion, almost all paths* $L^{(3)}(\omega)$ *have infinitely many double points.*

Proof. Let $0\leq a<b<c<\infty$. By Lemma 2, almost all (a,b)-paths $L^{(3)}_{a,b}(\omega)$ have a positive 3-dimensional capacity. By Lemma 1 and by the property (B_3) of Brownian motion, it is easy to see that $\Pr\{\omega \mid L^{(3)}_{a,b}(\omega)\cap L^{(3)}_{c,\infty}(\omega)\neq\theta\}>0$. From this it follows that there exists a real number d with $c<d<\infty$ such

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[2]) θ denotes the empty set.

[3]) Cf. e. g. R. Nevanlinna [8].

that $\Pr\{\omega \mid L^{(3)}_{a,b}(\omega) \cap L^{(3)}_{c,d}(\omega) \neq \theta\} = \delta > 0$. Let us put $a_k = a + kd$, $b_k = b + kd$, $c_k = c + kd$, $d_k = (k+1)d$, $k = 1, 2, \ldots$. Then $\Pr\{\omega \mid L^{(3)}_{a_k,b_k}(\omega) \cap L^{(3)}_{c_k,d_k}(\omega) \neq \theta\} = \delta > 0$, $k = 1, 2, \ldots$, and consequently (since the independence property (B_3) enables us to reproduce the standard argument of the zero or one law) $\Pr\{\omega \mid L^{(3)}_{a_k,b_k}(\omega) \cap L^{(3)}_{c_k,d_k}(\omega) \neq \theta$ for infinitely many $k\} = 1$.

Remark. It is easily seen from the proof that for all $0 \leqq a < b < \infty$ and for almost all ω the (a, b) path $L^{(3)}_{a,b}(\omega)$ has infinitely many double points. Thus if we count only the doube points for which $0 < t - s < \delta$ where δ is an arbitrarily small positive number, then again almost all paths $L^{(3)}(\omega)$ have infinitely many such double points. Similarly, for any arbitrarily large $\Delta < \infty$, almost all paths $L^{(3)}(\omega)$ have infinitely many double points with $t - s > \Delta$. (Of course, the probability that $L^{(3)}_{a,b}(\omega)$ have such double points is always smaller than 1; it is zero if $\Delta \leqq b - a$ and positive otherwise.)

§ 4. The 4-dimensional case.

Lemma 3. *Let $0 \leqq a < b < \infty$. Then for almost all ω, the (a, b)-path $L^{(4)}_{a,b}(\omega)$ of a 4-dimensional Brownian motion has zero 4-dimensional capacity.*

Proof. By the uniform Lipschitz property of Brownian motion [Lévy 5, § 52, pp. 166—173], there exist a finite constant M and a positive number $\delta(a, b, \omega)$ with $0 < \delta(a, b, \omega) < 1$ such that for almost all ω

$$|\mathbf{x}(t, \omega) - \mathbf{x}(s, \omega)| < M \sqrt{|t-s| \log 1/|t-s|} \tag{19}$$

holds for all s and t with $a \leqq s < t \leqq b$ and $t - s < \delta(a, b, \omega)$. Since the closed interval $[a, b]$ is a union of a finite number of closed intervals of length $< \delta(a, b, \omega)$, the property (C_3) of the capacity inplies that it is sufficient to show that $L^{(4)}_{a,b}(\omega)$ has zero 4-dimensional capacity whenever $b - a \leqq 1$ and (19) is satisfied for all s, t with $a \leqq s < t \leqq b$. Thus, by property (C_2) of the capacity it suffices to prove

Lemma 4. *If we put*

$$\lambda_p = \inf \frac{2}{p(p-1)} \sum_{1 \leqq i < j \leqq p} \frac{1}{|t_j - t_i| \log 1/|t_j - t_i|}, \tag{20}$$

where inf denotes the infimum for all p-systems $\{t_1, \ldots, t_p\}$ of real numbers t_i $(i = 1, \ldots, p)$ such that $0 \leqq t_1 < \ldots < t_p < 1$, then

$$\lim_{p \to \infty} \lambda_p = \infty. \tag{21}$$

Proof. Let N_m be the number of pairs (t_i, t_j) such that $2^{-m} \leqq t_j - t_i < 2^{-m+1}$, $m = 1, 2, \ldots$. Then

$$N_m = \frac{1}{2} p(p-1) \tag{22}$$

and

$$\frac{2}{p(p-1)} \sum_{1 \leqq i < j \leqq p} \frac{1}{|t_j - t_i| \log 1/|t_j - t_i|} \geqq \frac{2}{p(p-1)} \sum_{m=1}^{\infty} \frac{N_m}{2^{-m+1} \log 2^m} = \frac{1}{p(p-1) \log 2} \sum_{m=1}^{\infty} \frac{2^m N_m}{m}. \tag{23}$$

On the other hand, if we denote by $N_{m,k}$ the number of t satisfying $(k-1)2^{-m} \leqq t_i < k2^{-m}$, $k=1,\dots,2^m$, then

$$\sum_{k=1}^{2^m} N_{m,k} = p \tag{24}$$

and

$$\sum_{l=m+1}^{\infty} N_l \geqq \sum_{k=1}^{2^m} \frac{1}{2} N_{m,k}(N_{m,k}-1). \tag{25}$$

This follows from the fact that $(k-1)2^{-m} \leqq t_i < t_j < k2^{-m}$ implies $t_j - t_i < 2^{-m}$. Consequently, by the Schwarz inequality,

$$\begin{aligned} N_m^* &\equiv \sum_{l=m+1}^{\infty} N_l \geqq \frac{1}{2}\left\{\sum_{k=1}^{2^m} N_{m,k}^2 - \sum_{k=1}^{2^m} N_{m,k}\right\} \geqq \\ &\geqq \frac{1}{2}\left\{\left(\sum_{k=1}^{2^m} N_{m,k}\right)^2 \Big/ 2^m - \sum_{k=1}^{2^m} N_{m,k}\right\} = \frac{1}{2}\left(\frac{p^2}{2^m} - p\right) \geqq \frac{p^2}{2^{m+2}}, \end{aligned} \tag{26}$$

where the last inequality holds for those m which satisfy $2^{m+1} \leqq p$, i.e. for $m \leqq m_p \equiv \left[\frac{\log p}{\log 2}\right] - 1$.

Consequently, by Abel's transformation, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{2^m N_m}{m} &= \sum_{m=1}^{\infty} \frac{2^m(N_{m-1}^* - N_m^*)}{m} = 2N_0^* + \sum_{m=1}^{\infty}\left(\frac{2^{m+1}}{m+1} - \frac{2^m}{m}\right) N_m^* \geqq \\ &\geqq \sum_{m=2}^{\infty} \frac{m-1}{m(m+1)} 2^m N_m^* \geqq \frac{1}{3}\sum_{m=2}^{\infty} \frac{2^m N_m^*}{m} \geqq \frac{1}{3}\sum_{m=2}^{m_p} \frac{2^m}{m}\frac{p^2}{2^{m+2}} = \\ &= \frac{p^2}{12}\sum_{m=2}^{m_p}\frac{1}{m} \geqq \frac{p^2}{12}[\log(m_p+1) - \log 2] \geqq \frac{p^2}{12}(\log\log p - 2\log 2) \end{aligned} \tag{27}$$

which, together with (20) and (23), imply

$$\lambda_p \geqq \frac{\log\log p}{12\log 2} - \frac{1}{6} \longrightarrow \infty \tag{28}$$

as $p \longrightarrow \infty$.

From this it is easy to deduce our last result:

Theorem 2. *In a 4-dimensional Brownian motion, almost all paths $L^{(4)}(\omega)$ have no double points.*

Proof. In view of (B_3), it suffices to show that, for any rational numbers a, b, c, d, with $0 \leqq a < b < c < d < \infty$, we have $\Pr\{\omega \mid L_{a,b}^{(4)}(\omega) \cap L_{c,d}^{(4)}(\omega) \neq \theta\} = 0$. But this last fact is an easy consequence of Lemmas 1 and 3.

Bibliography.

1. J. L. Doob, Stochastic processes depending on a continuous parameter, *Transactions American Math. Soc.*, **42** (1937), pp. 107—140.
2. O. Frostman, Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions, *Meddelanden Lunds Univ. Mat. Sem.*, **3** (1935).

3. S. KAKUTANI, On Brownian motion in n-space, *Proceedings Academy Tokyo,* **20** (1944), pp. 648–652.
4. S. KAKUTANI, Two-dimensional Brownian motion and harmonic functions, *ibidem,* **20** (1944), pp. 706–714.
5. P. LÉVY, *Théorie de l'addition des variables aléatoires* (Paris, 1937).
6. P. LÉVY, Le mouvement brownien plan, *American Journal of Math.,* **62** (1940), pp. 487–550.
7. P. LÉVY, *Processus stochastiques et mouvements browniens* (Paris, 1948).
8. R. NEVANLINNA, *Eindeutige analytische Funktionen* (Berlin, 1936).
9. G. PÓLYA—G. SZEGŐ, Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen, *Journal für die reine und angewandte Math.,* **165** (1931), pp. 4—49.
10. N. WIENER, Generalized harmonic analysis, *Acta Math.,* **55** (1932), pp. 117—258.

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